**Manifestly Covariant Electrodynamics**

**An Aside on Lorentz Transformations of Fields**

Just want to echo some comments made apropos classical fields, in the SR dust/fluids file. That is, fields like φ(x,t), **A**(x,t), **E**(x,t), etc., do not transform simply by replacing {x,t} with {x´,t´}. For instance, in a frame where the charges are stationary, there would be no **B**, but in another reference frame there would **B** (ha!). So the fields do not transform covariantly in the sense of ‘same functional form’. But rather in the sense of ‘constructed the same way’. We use ME to construct the fields of the charges according to their perceived densities and velocities in each frame.

**Manifestly Covariant Force Law**

Just like in classical mechanics, we have to incorporate the findings of Special Relativity into EM. But the good news is that it is already in there. Basically, the **E**, **B** fields, etc., are already moving fast enough in our daily experience for us to have gotten the correct equations to begin with. So ME are correct in any reference frame. They are already Lorentz invariant – though not *obviously/manifestly* so. In order to put them in ‘manifestly’ covariant form, we want to write them in terms of space-time vectors, rather than just space vectors. And to improve the aesthetics, I’m going to use fake ‘Gaussian’ units (see Units file) where we set μ0 = 4π and ε0 = 1/4π, which makes c = 1 as well. These fake units have the virtue of making length and time dimensionally equivalent, and ridding us of pesky μ0, ε0 factor (replacing them with 4π and 1/4π respectively). And we can always convert our fake unit results back into real SI units (again, see file, or Appendix). So we’ll start with the Lorentz force law,



where τ is the proper time related to t via dt = γdτ. And we also have the work-energy equation.



U is the energy of the particle. Combining the two together, we have:



Now that we have things expressed in terms of the 4-velocity, and since it enters in linearly, we should be able to factor out the 4-velocity…so we’ll write out the RHS as:



And so we can combine these equations into:



Where (implicit summation over indices),



By the quotient rule,must be a four tensor (see Tensor file) and so now we know implicitly how the fields must transform under a velocity transformation, namely (implicit summation over repeated indices),



Let’s consider another way to come to the same conclusion – one which will give us another formula for F. First let’s start with the Lorentz gauge condition that:



If we define,



we can write this as:



Now  is a vector since its components transform like one, since:



and so by the quotient theorem is a vector. And now that we know how transforms, we implicitly know how the fields transform. To facilitate this, we can put  in terms of …So recall again,



We’ll note that the blue guys, which comprise index values α = {1, 2, 3}, β = {1,2,3} (index 0 refers to first row, column), involve strictly the magnetic field. And turns out we can write, for these guys values, where i and j run over indices {1, 2, 3} (and note implicit summation over repeated indices)



where εijk is the Levi-Cevita symbol,



And we can write as:



where ∂i = ∂/∂xi and ∂i = -∂/∂xi. We might therefore guess that:



and we would be correct as you might sometime explicitly verify. This tells us that F is the outer product of two tensors, and so is itself a Tensor (see Tensor file again). So in any event, this again explicitly tells us how **E**, and **B** transform from one reference frame to the next.



**Manifestly covariant Maxwell’s equations**

Now we’d like to put Maxwell’s equations in a manifestly relativistically invariant form. First, let me emphasize that they are already relativistically invariant. This fact was discovered by H. Lorentz (around 1875 or something) while studying the coordinate transformations under which Maxwell’s equations were invariant. So people knew Maxwell’s equations were invariant w/r to a Lorentz transformation way before they knew the significance of this fact. In any event, we want to put the equations in a manifestly invariant form by writing the equations in terms of space-time vectors, rather than just space vectors. We’ll start with the equations themselves, in ‘Gaussian’ units.



First define the space-time current:



(see Classical Mechanics folder on space-time vectors) This is a legitimate space-time vector since it obeys the continuity equation:



and so by the quotient theorem, since  is a space-time vector, and 0 is a scalar,  must be a space-time vector. So then we can write the Maxwell equations as:



Now let’s just consider the inhomogeneous ones and introduce the potentials.



Now add the two together – remember this is vector addition though.



So finally, we can write:



So we can write Maxwell’s equations (or at least the inhomogeneous ones) as, in space-time vector notation



**General transformation laws**

Consider the general case where we have some EM field in one frame. What will be the EM field in another frame moving with velocity **v** (say in the x-direction)? We would carry out, using a little help from Mathcad:



So this amounts to:

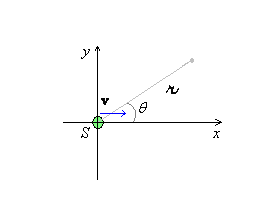


I think the discrepancy with Griffiths is that he’s using MKS units still. And we can write these as:

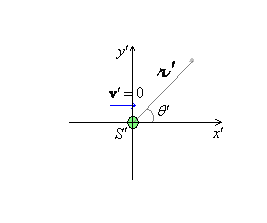


**Electric, Magnetic field of a moving point charge reprised [back to MKS units]**

The Lorentz transformation provides us an alternative way to calculate the field due to a moving charge. Consider a charge moving with velocity **v**. What is the field a distance r away?



To ascertain, we first jump to a reference frame which is stationary with respect to the charge S′. Note that as a consequence of jumping to this frame, length contraction will change the angle θ′ that the point seems to make with our location, and as such will change the distance, etc.



In this stationary frame, the field can be written as:



And F is:



Well, that’s in our fake Gaussian units. What is it in MKS units? Well see the Appendix. But it will be, in general:



And in particular,



Now let’s transform back to the S reference frame. This will require a Lorentz transformation Λ(**β** = -**v**/c), and so we’ll have:



which simplifies to:



So then we can read off the components:



This is consistent with the general analysis we did above. But not the same formula we found last time. This is because our present formula is in terms of the coordinates of the S′ reference frame where the charge is stationary. We want to put the expression in terms of the coordinates of the S reference frame where the charge is moving with velocity **v**. The coordinate transformation is, again **β** = -**v**/c.



which works out to:



Now we want our coordinate systems to cross when t = 0, and that is when we’re making our measurement. So we’ll set t = 0. Then we have:



Putting it together we have:



and so we have:



which is precisely what we had before. Now for the magnetic field we have:



which enables us to write:



just as we found before.

**Example**

Consider a charge, q, and another charge q′ moving with speed **v′** a distance r away. What is the force on q′ in the q reference frame? The force is:



Consider now that we’re moving with the charge q′ at speed **v′**. Then from this perspective it is the charge q which is moving, with speed. Well, we get the same result, ‘cause in this reference frame we see an electric and magnetic field, but since q′ is stationary in this frame, only the **E** field exerts a force and it will be the same force as above.

**Example: How this is consistent with what we’d expect from relativity**

Magnetic fields are a relativistic effect of moving charged particles. Consider a positively charged segment of length L and charge density λ. If we’re at rest with respect to the rod, then we observe an electric field of:



Now let us accelerate to a velocity **v** w/r to the rod. Due to length relativistic length contraction we will see the rod shrink by a factor of γ. And we will see it move backwards with velocity –**v**. So in our reference frame we will see the fields,



which is consistent with the relativistic transformation laws since we have,



**Appendix: Converting Back to MKS units**

So let’s consider what a lot of these guys would look like in MKS units. First we’ll look at the 4-velocity and 4-momentum:

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For the record, in both unit systems, the basis vectors have no units. The four velocity, as expressed in Gaussian units, is dimensionally consistent since **v** also is unitless (because in Gaussian units length and time have same dimension). But not so in MKS units. Then the spatial part of the 4-velocity has units of velocity, i.e., L/T, while the time component is still unitless. As discussed in the Units file, this is because it’s missing requisite factors of (4πε0) and (μ0/4π). And if we restored the appropriate powers of these guys, the time-like part would have units of velocity. We can brute force figure out what these must be. We need:



So we need to multiply the time part by the factor,



So that’s why we did that up there. Similar reasoning would lead to us dividing the time part of the 4-momentum by c, to convert to MKS. The 4-current will need a ‘c’ on the time part in MKS, to make the charge density ρ dimensionally consistent with current, **j**. And since in MKS, units of φ = ML2/T2C and of A = ML/CT (see Units file), we see that for the four-potential needs division by c on the time part to make it dimensionally consistent with **A**. So,



What about **F**? To make the EM tensor dimensionally consistent, we will divide the E’s by c. Could’ve alternatively multiplied the B’s by c. Which should we do? Kind of a matter of preference I guess. But seems that people prefer the former approach, which gives F the dimensions of magnetic field, B.



The 4-gradient will have to change to:



We could’ve alternately multiplied the ∇ by c, but we’d rather keep the spatial part of the 4-gradient equal to the gradient operator. Now the Lorentz force equation would need to be changed to:



(so it doesn’t change) because the LHS clearly has dimensions of force, and RHS does as well already, since **F** has dimensions of **B**, and **υ** dimensions of v, and we have that q with dimensions of charge. Now what about that Maxwell’s equation? Since the 4-gradient has dimensions of gradient, **F** has dimensions of **B**, and j has dimensions of current density, thinking back to the ∇×B = μ0j equation, we can see that we need 1 in the form of μ0 on the RHS to make the units work out. So we multiply by μ0/4π.



And last, those E, B transformation equations will go as, just making them dimensionally consistent:



By the way, in Gaussian units, (**β** = **v**/c) = **v**.

**A little Special Relativity Review**

Little review – this is covered in more detail in the Classical Mechanics folder where Special Relativity is discussed, and the math is covered in a lot more detail in the Tensor file. And by the way, lots of implicit summation over repeated indices here (Einstein summation convention) Alas, it is convention to define the space-time metric, a little differently than in classical mechanics. Now we have:



This changes how covariant and contravariant basis vectors and components are related. The space-time basis vectors are:



Inner (dot) products between basis vectors are defined by the metric, which is not Euclidean. So we don’t get what we’d necessarily expect. What we do have is:



Where δαβ is a diagonal matrix (1,1,1,1). And we’ll note in particular that (no implicit summation) , where i = 1,2,3. The space-time (maybe should say time-space) position vector is given by:



The x0,1,2,3 are the contravariant components, and x0,1,2,3are the covariant components. The space-time velocity vector is given by:



And υ0,1,2,3, υ0,1,2,3 are the contravariant, covariant components of the space-time velocity vector. The space-time momentum vector is given by:



And p0,1,2,3 are the contravariant components, p0,1,2,3 are the covariant components. Also note p0 = p0 = mc2γ/c = U/c, where U is the particle’s energy. And we can generalize beyond vectors, to tensors (the EM field will turn out to be involved in one of these, shorrtly).



where Tαβ are called the contravariant components of the tensor . Or we can write it as:



where Tαβ are called the mixed components of the tensor . And could write as:



where Tαβ constitute a different set of mixed Tensor components. And finally we can write it as:



where Tαβ are called the covariant components of the tensor. The covariant and contravariant components of a vector are all related via the metric, which functions as a raising/lowering operator (again implicit summation over repeated indices).



These relations can be interpreted as matrix multiplications (see Tensor file). So dot product between two vectors is:



Can also write this as:



And could say likewise:



Could do likewise with dot products between vectors and tensors. Finally, the coordinate transformation matrix (for particle moving along x axis with speed v = βc) is as usual,



And this tells us how the components of a vector or tensor will change from the unprimed/stationary (we’ll say) coordinate system, to the primed (moving with speed v) coordinate system. The coordinate transformation work like this:



These can be written as matrix equations too.