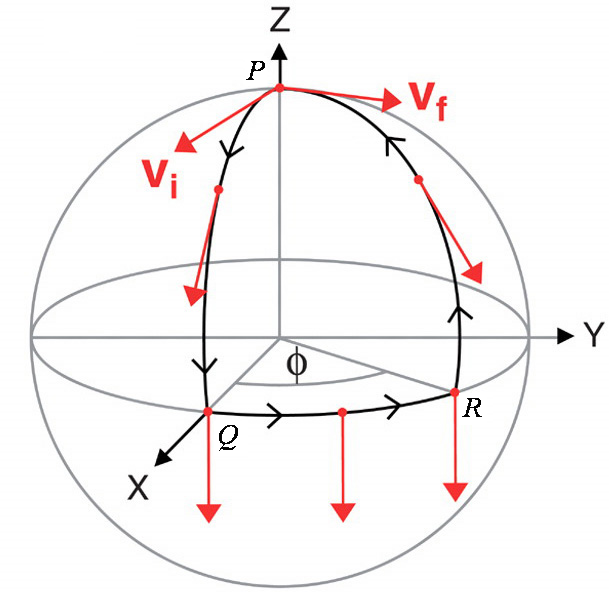
**Tensors and stuff**

(note we’re using Einstein summation convention, implicitly; maybe see Appendix at bottom)

**7. Curvature**

Finally let’s return to the topic at the top of the page. We said that the metric tensor encoded information about the curvature of the space. We would like to investigate this matter a little more thoroughly and develop some mathematical formula which quantifies the curvature of a space. First let’s make a distinction between extrinsic and intrinsic curvature. A surface has extrinsic curvature if it isn’t flat in some higher D Cartesian space. For instance, the 2D cylinder surface has extrinsic curvature w/r to its 3D environment. But it actually has no intrinsic curvature b/c the surface can be completely ‘unwrapped’ flat in 2D. Perhaps a criterion would be, ‘if a right angled closed path looks like a rectangle, then you’re flat; otherwise not’. On the contrary, a 2D spherical surface has extrinsic curvature, and it also has intrinsic curvature because it cannot be unwrapped into a 2D flat surface w/o distortion. We made the point at the top of the page that the Pythagorean theorem should hold, for any size triangle, on an intrinsically flat (though perhaps extrinsically curved) surface, but not on an intrinsically curved surface. So for instance the Pythagorean theorem would hold for any triangle on a cylinder, but not on a sphere. We will only focus on the notion of intrinsic curvature since whether the hyper-surface we live on is embedded in a higher dimensional volume is not known. The notion of intrinsic curvature is elucidated below:

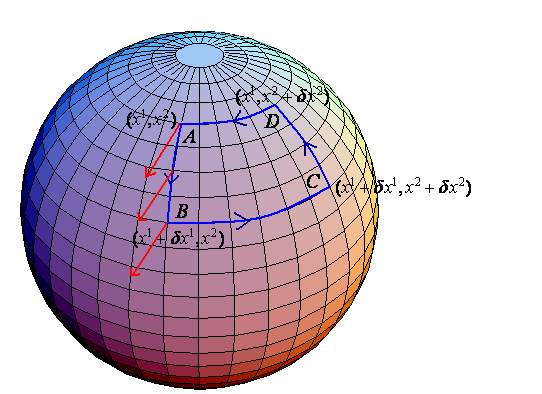


Consider a vector at P. If we pick the vector up and transport it to Q, without changing its local orientation, then to R in the same way, and then back to P in the same way, we see that the orientation of the vector has changed from where it was at the beginning of the path. This happens precisely because the surface is intrinsically curved. If we try the same experiment on a flat surface, even in polar coordinates, then the vector will still be parallel to itself when it gets back to P. Note that ‘w/o changing its local orientation’ doesn’t mean keeping the components of the vector the same between two differentially displaced coordinates. It does mean keeping it parallel to itself between two differentially displaced coordinates.

How can we mathematically define this? Since we can differentiate basis vectors, that measn we can put the difference between basis vectors at xα+dxα in tems of the ones at xα. And that means we can put the basis at xα+dxα in terms of the basis at xα. And so then we can determine whether a vector at xα+dxα is the same as the vector at xα. So to that end, let’s consider a local vector field **V**(xα), with a vector at every coordinate in the vicinity of xα. Then our vector field will consist of parallel transported vectors if the vectors do not change anywhere in the neighborhood of xα. The gradient measures change in any direction. So we’d need the gradient of our vector field to be zero.



So this is the equation satisfied by a vector parallel transported along a curve. Now let’s use this equation to parallel transport our arbitrary vector V around the entire square loop below and then see what we get.



Let’s start with a vector **V**, located at coordinates A = (x1, x2). We will parallel transport this vector to B = (x1+δx1, x2), and then to C = (x1+δx1, x2+δx2), and to D = (x1, x2+δx2), and then back to A = (x1, x2) – in other words counter clockwise around the square. Then we will look at the difference between the transported vector and its original self. The difference will be a measure of the curvature of the surface. I guess we’ll focus on the *components* of the vector, since these would form a scalar field, which should definitely obey that gradient-integral equation ∫d**r**·∇φ = φ(b) – φ(a) [and not sure the *vector* itself would]. So we have:



Simplifying this expression we have:



where in the last we use the rectangle approximation of the integrals. Next we execute the derivatives…



Now let’s generalize our expression to transport along some general curve, not restricted to the coordinate axes. Then we have:



And we take the term in brackets to define our curvature. It is called the Riemann curvature tensor:



So we would understand this as the change in the αth component as a fraction of its βth component, as we make a closed loop along the μ and ν directions. Another way to define the curvature tensor is by looking at the commutator of covariant differentiation. Consider the covariant derivative in the direction *a*, followed by the derivative in direction *b* (both *a* and *b* are along coordinate lines)



Now look at the reverse:



and subtract the two:



and so we have:



which defines the commutator of the derivatives of a vector **V** with the curvature tensor in that vicinity. I think we can be sure that Rncab is a tensor, here, because if we had summed over *a* and *b*, then we would’ve gotten a tensor for sure because ∇ is a tensor (when combined with its basis vectors), and so without the summation, we just have that Rncab forms the components of a tensor.

Now let’s look at a few identities that the Riemann tensor satisfies. In order to make the analysis more tractable, we will go to a coordinate system which is locally flat, wherefore Γabc = 0, and first derivatives of the metric tensor are also 0 consequently (since gαβ = **e**α·**e**β and the first derivatives of the **e**’s are zero ) at that point. Recall that we can go to a locally flat coordinate system because of our blue comments at the top of the page. In that case, we see that:



(the Γαβ[ν,μ] term isn’t zero because it’s a *second* derivative) And now we fill in the Christoffel identity to get:



Lowering the index on the Riemann tensor we have:



From here we may observe a few identities:



So it is anti-symmetric w/r to its first two indices, and last two indices. But it is symmetric w/r to its ‘block’ indices. Now one might suppose that these identities hold only in the locally inertial reference frame, but in fact they hold in all reference frames since we may change to any coordinate system at will. On to the Bianchi identity. We take the derivative of the Riemann tensor w/r to xλ in a locally flat coordinate system:



and from this one can show that:



which generalizes in an arbitrary coordinate system to:



**Example: Rαβμν = 0 for cylindrical manifold**

Show that the Riemann tensor is identically zero for a cylindrical coordinates in Euclidean space. Well,



Now,



and



So the Christoffel symbol is:



Next we must calculate the derivative of the Christoffel symbol, which is:



and now the Riemann tensor is:



The fact that we get 0 for all elements indicates that the polar geometry is intrinsically flat. This is evinced by the fact that we can unroll the surface of a cylinder into a flat sheet.

**Ricci tensor and scalar**

Now let’s discuss a related tensor – the Ricci tensor. The Ricci tensor is a contraction of the Riemann tensor.



It is essentially the only non-trivial contraction possible since the other two contractions gives zero due to the anti-symmetry of Rαβγδ. Let’s try to work this out in terms of the Christoffel symbol.



Regardless, a further contraction gives us the Ricci scalar,



**Example: Rαβ = R = 0 for polar metric**

So we already know that Rαβγδ = 0 which implies that Rαβ and R are both zero as well. But let’s go ahead and make the calculation anyway, directly from the Γ’s.



and Rαβ is given by:



and consequently R = gαβRαβ = 0 of course.