**Tensors and stuff**

(note we’re using Einstein summation convention, implicitly; maybe see Appendix at bottom)

**5. Derivatives of Basis Vectors**

Now we want to begin a discussion on differentiation. We’ll start with the basis vectors. And we’ll look at an application along the way. I want more lines in here before starting the next section because that looks nicer. This should be enough I think. Bare minimum really. Well maybe one more line? Yeah that looks nicer.

**5a. Derivatives of basis vectors and Christoffel symbols**

The derivative of the basis vector, such as it is, must be expressable in terms of the basis vectors at that point. And so we may denote the requisite linear combination with the so-called Christoffel symbol, Γ.



Now it is generally only in flat geometries (i.e. Euclidean ones) that this linear combination exists. In curved geometries, we may run into problems. For instance, if we confine ourselves to the surface of a cylinder, then ∂**e**φ/∂φ will involve **e**r, which does not exist in that geometry. I believe that in such cases where basis vectors normal to our surface are invoked, we just set that term to zero [seen people do this at least]. Seems kind of ad hoc. But it would make sense because a cylindrical surface is in some sense flat, and so its basis vectors should *not* change. This issue doesn’t arise when we use Γ’s representation in terms of the metric [see below], as the metric will only involve the d.o.f. available to us. Moving on…note it follows that:



Does Γ form the components of a tensor? Well,



So it doesn’t. Let’s invert this:



and now let’s switch the greek and roman letters.



These are the same of course, though it doesn’t obviously look like it. Hmmm. On a side note, observe that ∂u´α/∂u´β ≠ δαβ unless the coordinate system is orthogonal. Well I actually I think that identity is preserved regardless. Note that Γ is symmetric in its lower indices since:



so,



There are some formulas for Γcab but mostly one just has to work out the derivatives of **e**a to figure out what it is. One useful formula that we’ll use is for a certain contraction Γaab. We can work it out as follows.



and so (almost) finally,



Now there is a useful result about such matrix products, namely that for any matrix A:



Could say ∂lnA/∂x = ∂ln|A|/∂x – just saying. Using this result we can say:



We will use this result a little later. We can actually write the entire Γ in terms of the metric, as it turns out. We do it like this. Consider:



Permuting the free indices we also get:



Adding the last two and subtracting the first we have:



From which we infer that:



And so we have that:



Note the indices are just cyclical permutations of the first, with the appropriate sign change. Γkij is called the Christoffel symbol of the 1st kind, incidentally. Γkij is called the Christoffel symbol of the 2nd kind – and remember they are different fundamentally because the symbols aren’t tensor components. Now let’s consider derivatives of the contravariant basis vectors. These are:



Perhaps can start from,



So there we go.



**Example: Christoffel symbols for polar metric**

First let’s note how to handle matrix multiplication – when written out. Suppose have two matrices A and B.



which I’m going to write as:



Then the matrix C defined via the equation below, is:



So note that we can multiply the matrices together distributively, like we would any set of numbers. But (third line) we only keep products that satisfy the requisite form in the multiplication expression, i.e. the ones with the same k-value. And in the fourth line we have to transpose the indices in the second ( ) because of the form C is in.

The polar metric is:



and so we have:



Observe that it is symmetric in the lower index. How much does the radial basis vector change when we go half-way around a circle? This would be:



well that’s true.

**Example: Christoffel symbols on surface of sphere**

Let’s consider the basis vectors on the surface of a sphere. What is the Christoffel symbol? Well the metric is:



(remember there is no r derivative b/c r is a constant here) and so the Christoffel symbol is:



and so for instance,



Now let’s look at it a different way. We’ll describe the spherical surface using Cartesian vectors. Then the basis vectors are [just **e**θ and **e**φ, but I’ll include **e**r for later use]



And let’s look at d**e**φ/dφ. It is:



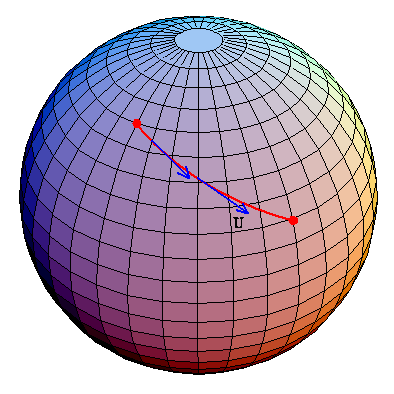
Now, can we put this in terms of **e**φ and **e**θ themselves? Nope. Nonetheless, we can express it in terms of these, neglecting the **e**r projection, as follows, using a resolution of unity...



and so we get the same result.

**5b. Geodesics**

A geodesic is a curve along some geometrical surface that is locally straight. It is the shortest distance between two points on that surface. We would like to determine the equation for this curve.



So the key characteristic of this curve uα(s) is that the tangent to the curve does not change direction. Now the vector tangent to the curve is:



and specifying that the tangent vector doesn’t change means that:



Note how this looks similar to the equation that the acceleration of the particle is 0. Anyway, working this out we have that:



where we have changed indices in the first term. So now we can write our geodesic equation as:



We’ll note that in Euclidean space Γ = 0 of course, and we just get a straight line equation.